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NASA PROJECT APOLLO WORKING PAPER NO. 1005

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PRELIMINARY SURVEY OF RETROGRADE VELOCITIES REQUIRED FOR INSERTION INTO LOW LUNAR ALTITUDE ORBITS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

SPACE TASK GROUP

Langley Field, Va.

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Preliminary Survey of Retrograde Velocities Required for Insertion into Low Lunar Altitude Orbits

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SUMMARY

Closed lunar orbits are envisaged in the manned lunar mission program. The study described herein was undertaken to obtain an appreciation of the relevant fuel consumption requirements. The retrograde impulses necessary for establishing the orbits were assumed to occur at the point of closest approach of the main earth-moon trajectory; this point designated as the arrival position was restricted to a lunar altitude of 5,000 nautical miles or less. The orientation of the arrival position vector relevant to any coplanar radius vector is not constrained, however, and similarly the scalar value of the arrival velocity is unrestrained.

Since the arrival altitude is restricted to 5,000 nautical miles, the perturbing accelerations of the earth and sun are sufficiently small that the vehicle and moon essentially comprise an isolated two-body system. This is further discussed in the report.

Retrograde velocities are determined for any required pericynthion positions. If the pericynthion orientation requirement is relaxed, then a smaller retrograde velocity is in some cases possible. A comparison between minimum retrograde velocities and retrograde velocities necessary for stipulated pericynthion positions is given. Arrival velocities are correlated with earth insertion velocities for a feasible earth insertion position.

The equations developed for determining retrograde velocities for desired pericynthion positions are considered useful for estimating essential data for the preliminary planning of the manned lunar mission. Some graphical representation is included herein for immediate familiarization with possible conditions.

INTRODUCTION

The study described in this report was initiated by the desire for a preliminary assessment of various problems associated with local lunar orbits as envisaged in the manned lunar mission program. Since it was known that the lunar gravitational field would be dominant during low altitude orbits and two-body solutions would be applicable, it was decided to take advantage of this and use a closed solution approach. When satellites of a dominating gravitational field are studied, velocities relative to the nonrotating coordinate system, origin the gravity source, are treated as inertial in order that Newtonian laws may apply. As an example, planet velocities relative to the sun are considered inertial and yet the sun is thought to be moving in space. For a lunar satellite, a perfectly elliptical orbit cannot be achieved due to the movements of the sun, earth, and moon relative to the vehicle; however, for low altitude lunar orbits, the lunar gravitational field is dominant; hence near elliptical orbits may occur. Stability checks of the ellipticity were examined from the output of the Republic digital program noted in reference 1. The checks confirmed the validity of the approach taken.

Contained within this report is a method for determining the retrograde velocity necessary for establishing orbit with specific characteristics. It is anticipated that the plane of the vehicles lunar arrival velocity will, by guidance impulses, nearly coincide with the plane of the desired resultant orbit; hence, one constraint of this study was that both the arrival velocity and resultant orbit are coplanar. This report also indicates the correlation of arrival velocity with specified earth insertion conditions by means of a restricted three-body mathematical model.

It is possible in many cases to reduce the magnitude of the retrograde impulse if the restrictions on the pericynthion orientation are relaxed. Therefore, a method is presented whereby the minimum retrograde impulse can be determined for an orbit with a specified pericynthion radius with no constraint on orientation. The methods of determining minimum retrograde velocities and retrograde velocities for required response are thought to be useful in that they point the way to a comprehensive quantitative survey program for assessing fuel requirements for lunar orbits.

SYMBOLS

a	semimajor axis of the orbit, ft
C	Jacobian constant, nondimensional
D	distance from center of earth to center of moon, ft
e	Naperian logarithm base
G	universal gravitational constant, $ft^4/lb\ sec^4$
g_o	gravitational acceleration at earth's surface, ft/sec^2
h	altitude
h_a	altitude at apocynthion
h_p	altitude at pericynthion
I	specific impulse of fuel, sec
M	mass of earth plus mass of moon, slugs
M_e	mass of earth, slugs
M_o	mass of moon, slugs
r	distance from moon center to vehicle position, ft
r_A	apocynthion distance from moon center, ft
r_{bp}	distance from barycenter to vehicle, ft
r_{ep}	distance from earth center to vehicle position, ft
r_p	pericynthion distance from moon center, ft
R_e	distance from earth center to barycenter, ft
R_m	distance from moon center to barycenter, ft
t	time, sec
V	velocity referred to the rotating coordinate system, origin the barycenter, ft/sec
V_A	arrival velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_e velocity referred to the nonrotating coordinate system, origin the earth center, ft/sec

V_i velocity referred to the nonrotating coordinate system, origin the barycenter, ft/sec

V_m velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_o orbit velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_{oa} orbit apocynthion velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_{op} orbit pericynthion velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_R retrograde velocity, ft/sec

X,Y,Z rotating coordinates, origin the barycenter

X_e,Y_e,Z_e nonrotating coordinates, origin the earth center

X_i,Y_i,Z_i nonrotating coordinates, origin the barycenter

X_m,Y_m,Z_m nonrotating coordinates, origin the moon center

α difference between θ and $\bar{\theta}$, degrees

γ angle between vehicle velocity vector and local horizontal, degrees

δ fuel mass to gross mass ratio at commencement of burning for entry into local lunar orbit

$\bar{\delta}$ overall fuel mass to gross mass ratio required for both the entry into and exit from the local lunar orbit at the same point of orbit

ϵ eccentricity

θ angle between r and r_ρ degrees at instant of retro impulse burnout

$\bar{\theta}$ angle between r and ζ-axis, degrees function of time at instant of retro impulse burnout

μ	$\mu = \frac{M_o}{M}$, nondimensional
$1-\mu$	$1-\mu = \frac{M_e}{M}$, nondimensional
ρ_o	included angle between V_A and V_R, degrees
ω	rotational velocity of earth-moon system, radians/sec

ARRIVAL VELOCITY

In the manned lunar mission program, it is assumed that one of the mission's main objectives will be to make a close survey of the moon's terrain. For a detailed survey, this will require the vehicle to establish an orbit about the moon which will necessitate the application of a retrograde impulse. It is assumed for this study that the impulse will be applied at the instant of closest approach to the moon when the vehicles velocity vector is normal to the extension of the moon's radius. A conception of the complete mission is shown in figure (1).

The vehicle's lunar arrival velocity and position can be correlated with the earth insertion velocity and position by reference to a restricted three-body mathematical model. In this model the moon and earth are considered to rotate with constant radii and constant angular velocity about their common center of mass. The earth and moon are considered as point masses and the vehicle's mass is regarded as infinitely small in comparison. Further information concerning the characteristics of the restricted three-body mathematical model are found in reference 2.

The restricted three-body equations of motion as given in the rotating coordinate system with origin the barycenter are

$$\ddot{X} = \omega^2 X + 2\omega\dot{Y} - \frac{GM(1 - \mu)(X - X_1)}{r_{ep}^3} - \frac{GM\,\mu(X - X_2)}{r^3} \tag{1.1}$$

$$\ddot{Y} = \omega^2 Y - 2\omega\dot{X} - \frac{GM(1 - \mu)\,Y}{r_{ep}^3} - \frac{GM\,\mu\,Y}{r^3} \tag{1.2}$$

$$\ddot{Z} = \frac{-GM(1 - \mu)\,Z}{r_{ep}^3} - \frac{GM\mu\,Z}{r^3} \tag{1.3}$$

where $(X - X_1)$ is the distance along the X coordinate from the earth center to the particle and $(X - X_2)$ is the distance along this coordinate from the moon center to the particle.

By writing

$$W(X,Y,Z) = \frac{1}{2}\,\omega^2(X^2 + Y^2) + \frac{GM(1 - \mu)}{r_{ep}} + \frac{GM\mu}{r} \tag{1.4}$$

then the equations of motion become

$$\ddot{X} = \frac{\partial W}{\partial X} + 2\omega Y \tag{1.5}$$

$$\ddot{Y} = \frac{\partial W}{\partial Y} - 2\omega X \tag{1.6}$$

$$\ddot{Z} = \frac{\partial W}{\partial Z} \tag{1.7}$$

By multiplying the above by $2\dot{X}$, $2\dot{Y}$, and $2\dot{Z}$ respectively, adding, and integrating, Jacobi's integral is obtained, which is

$$(\dot{X})^2 + (\dot{Y})^2 + (\dot{Z})^2 = \omega^2(X^2 + Y^2) + \frac{2GM(1 - \mu)}{r_{ep}} + \frac{2GM\,\mu}{r} - C \tag{1.8}$$

or

$$C = \omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}} + \frac{2\mu GM}{r} - V^2 \tag{1.9}$$

If the earth insertion velocity and position are known, the integration constant C can be determined. Once C is determined, it is possible to calculate the scalar value of the lunar arrival velocity for any given arrival position. The transformation of the velocity in the rotating coordinate system, origin the barycenter, to the nonrotating coordinate system, origin the moon center, is

$$\vec{V}_m = \vec{V} + \vec{\omega} \times \vec{r} \tag{1.10}$$

The relationship between the velocities in the different coordinate systems is illustrated in figure 2. For a comparison between earth insertion velocities and lunar arrival velocities, refer to figure 3. It should be noted that the vehicle is confined to the earth-moon plane which contain the axes X, Y; X_i, Y_i; X_e, Y_e; X_m and Y_m. The correlation between arrival velocity and earth-insertion velocity is shown for the condition of minimum earth-moon distance. The vehicle's arrival velocity is in the $(-\vec{\omega} \times \vec{r})$direction.

It is shown below that for a given C and arrival position vector, regardless of the orientation of the position vector, that the lunar arrival velocity is approximately constant. This assertion must be qualified by restricting the altitude to the range considered in this study. The velocity relationship is as follows:

$$V^2 = \omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}} + \frac{2\mu GM}{r} - C \tag{1.8-A}$$

For convenience, equation (1.8-A) is rewritten such that the terms, independent of the orientation of $\vec{r}$, appear on the left-hand side.

$$V^2 - \frac{2\mu GM}{r} + C = \omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}} \tag{1.8-B}$$

Upon utilizing the law of cosines, the following is evolved

$$\left[\omega^2 r^2_{bp} + \frac{2(1 - \mu)GM}{r_{ep}}\right] = \omega^2\left[\left(R_m^2 + r^2\right) + 2R_m\left(X - R_m\right)\right] + \frac{2(1 - \mu)GM}{\left\{\left[\left(R_e + R_m\right)^2 + r^2\right] + 2\left(R_e + R_m\right)\left(X - R_m\right)\right\}^{\frac{1}{2}}} \tag{1.11}$$

$$\left.\begin{array}{l} \text{Since } \left(R_m^2 + r^2\right) >>> 2R_m\left(X - R_m\right) \\ \text{and } \left[\left(R_m + R_e\right)^2 + r^2\right] >>> 2\left(R_m + R_e\right)\left(X - R_m\right) \end{array}\right\} \begin{array}{l} R_m >>> r \\ \text{and if } \left|X - R_m\right| \leqslant r \end{array}$$

Therefore

$$\left[\omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}}\right] \approx \omega^2\left(R_m^2 + r^2\right) + \frac{2(1 - \mu)GM}{\left[\left(R_m + R_e\right)^2 + r^2\right]^{\frac{1}{2}}}$$

This allows equation (1.8-A) to be modified to an excellent approximation for the range of altitudes studied.

$$V^2 \approx \omega^2\left(R_m^2 + r^2\right) + \frac{2(1 - \mu)GM}{\left[\left(R_m + R_e\right)^2 + r^2\right]^{\frac{1}{2}}} + \frac{2\mu GM}{r} - C \tag{1.8-C}$$

Consequently it can be seen from equation (1.8-C) that if r is constant, then the orientation of r will not affect the value of V.

The trend of the arrival velocity with increase of altitude above the lunar surface may be more clearly understood by the following considerations. Substitution of equation (1-11) into equation (1.8-A) yields

$$V^2 = \omega^2\left[R_m^2 + r^2 + 2R_m\left(X - R_m\right)\right] + \frac{2(1-\mu)GM}{\left[\left(R_e + R_m\right)^2 + 2\left(R_e + R_m\right)\left(X - R_m\right)\right]^{\frac{1}{2}}} +$$

$$\frac{2\mu GM}{r} = C \tag{1.8-D}$$

Differentiation of this equation with respect to r yields

$$\frac{\partial V^2}{\partial r} = 2\omega^2 r - \frac{2(1-\mu)(GM)r}{\left[\left(R_e + R_m\right)^2 + r^2 + 2\left(R_e + R_m\right)\left(X - R_m\right)\right]^{\frac{3}{2}}} - \frac{2\mu GM}{r^2} \tag{1.12}$$

however,

$$\left\{2\omega^2 r - \frac{2(1-\mu)(GM)r}{\left[\left(R_e + R_m\right)^2 + r^2 + 2\left(R_e + R_m\right)\left(X - R_m\right)\right]^{\frac{3}{2}}}\right\} \lll \frac{2\mu GM}{r^2}$$

Therefore, it is possible to reduce equation (1.12) to the approximation

$$\frac{\partial V^2}{\partial r} \approx \frac{-3.5 \times 10^{14}}{r^2} \text{ ft/sec}^2 \tag{1.13}$$

This expression gives a fine approximation for the trend of the arrival velocity with altitude. Although this approximation applies for a velocity V, which is referred to the rotating axis system with origin the barycenter, it is shown below that with little loss of accuracy that V may be regarded as V_A . Given that

$$V_A^2 = V^2 \pm 2\omega r V + \omega^2 r^2 \tag{1.14}$$

then

$$\frac{\partial V_A^2}{\partial r} = \frac{\partial V^2}{\partial r} + \frac{\partial}{\partial r} (\omega^2 r^2) \pm \frac{\partial}{\partial r} (2\omega r V) \tag{1.15}$$

which yields

$$\frac{\partial V_A^2}{\partial r} = \frac{\partial V^2}{\partial r} \pm \frac{\omega r}{V} \frac{\partial V^2}{\partial r} + 2\omega^2 r \pm 2\omega V \tag{1.16}$$

and upon collecting terms

$$\frac{\partial V_A^2}{\partial r} = \frac{\partial V^2}{\partial r} (1 \pm \frac{\omega r}{V}) + 2\omega(\omega r \pm V) \tag{1.17}$$

By inserting a relevant range of values into the equation it is found that

$$\frac{\omega r}{V} \lll 1$$

and

$$2\omega(\omega r \pm V) \lll \frac{\partial V^2}{\partial r}$$

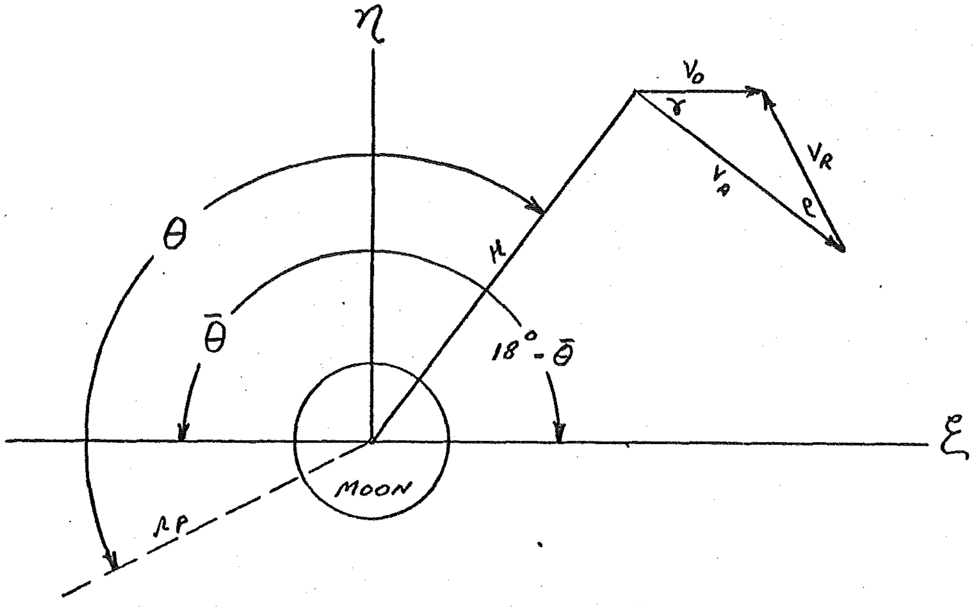
therefore, the following approximation is acceptable:

$$\frac{\partial V_A^2}{\partial r} \approx \frac{\partial V^2}{\partial r} \approx \frac{-3.5 \times 10^{14}}{r^2} \quad \text{ft/sec}^2 \tag{1.18}$$

DETERMINATION OF RETROGRADE VELOCITIES

This section contains a method for determining instantaneous retrograde velocities within certain constrained conditions. For the method described herein, the plane of action is defined as the plane containing the moon center and the arrival velocity vector. The retrograde impulses will be applied in this plane, and therefore, the resulting orbits will be in this plane. Also, since the retrograde impulse is to be initiated at the point of closest approach to the moon, then the arrival velocity vector will be normal to the position vector. The radius of curvature of the resultant trajectory will never be decreased immediately after the application of retrograde impulse if the orbit is elliptical. For nonelliptical orbits, the radius of curvature will be decreased; however, an alternative solution for the retrograde impulse determination is required as will be brought out in the text.

The method for determining an orbit with a required pericynthion radius and orientation is given below. The following sketch depicts the vehicle's arrival at the vicinity of the moon where the plane of arrival is denoted as the ζ, η plane.



By the law of cosines

$$V_R^2 = V_A^2 + V_o^2 - 2V_oV_A \cos \gamma \tag{2.1}$$

where

$$\cos \gamma = \frac{r_p V_{op}}{rV_o} \tag{2.2}$$

and

$$V_o^2 = GM_o\left(\frac{2}{r} - \frac{1}{a}\right) \tag{2.3}$$

Substitution of equations (2.2) and (2.3) yields

$$V_R^2 = \frac{GM_o(2a - r)}{ar} + V_A^2 - \frac{2V_A r_p V_o}{r} \tag{2.4}$$

It is seen from the relationships as outlined in the appendix that

$$a = r_p\left(\frac{\frac{r_p}{r} - \cos\theta}{2\frac{r_p}{r} - 1 - \cos\theta}\right) \tag{2.5}$$

It also follows from the relationships in the appendix that

$$V_{op} = \left\{GM_o\left[\frac{1 - \cos\theta}{r_p\left(\frac{r_p}{r} - \cos\theta\right)}\right]\right\}^{\frac{1}{2}} \tag{2.6}$$

Substitution of equation (2.5) into (2.3) yields

$$V_o^2 = \frac{GM_o}{r_p}\left[\frac{r}{r_p}\left(\frac{1 - \cos\theta}{1 - \frac{r}{r_p}\cos\theta}\right) + 2\left(\frac{r_p}{r} - 1\right)\right] \tag{2.7}$$

now let

$$\frac{r_p}{r} \equiv \phi \tag{a}$$

$$\frac{GM_o}{\phi r_p} \equiv A^2 \tag{b}$$

$$\frac{1 - \cos\theta}{1 - \frac{1}{\phi}\cos\theta} \equiv X \tag{c}$$

$$\frac{2GM_o}{r_p}(\phi - 1) \equiv K \qquad \text{(d)}$$

Substitution of identities a, b, c, and d allows equation (2.7) to be rewritten

$$V_o^{\,2} = A^2 X + K \tag{2.8}$$

Equation (2.1) may now be written as

$$V_R^{\,2} = V_A^{\,2} + A^2 X + K - 2V_A \phi A X^{\frac{1}{2}} \tag{2.9}$$

Equation (2.9) is the general equation which determines the retrograde velocity for the required conditions.

Transposing equation (2.2) yields

$$\gamma = \text{arc}\ \cos\left(\frac{r_p}{r}\right)\left(\frac{V_{op}}{V_o}\right) \tag{2.2-A}$$

By determining γ then ρ can be calculated from the law of sines. Thus

$$\rho = \text{arc}\ \sin\left(\frac{V_o}{V_R}\sin\gamma\right) \tag{2.10}$$

Equations (2.9) and (2.10) determine the value and sense of the required retrograde velocity. However, it should be noted that the resultant trajectory could represent any type of conic orbit depending on the arrival conditions and the required characteristics of the resulting trajectory. It is assumed that an elliptical orbit is desired; however, it does not necessarily follow that the retrograde impulse will yield an elliptical orbit. The classification of the conic orbit can be found by determining the eccentricity where

For an elliptical orbit $0 < \epsilon < 1$, for a parabolic orbit $\epsilon = 1$, and for a hyperbolic orbit $\epsilon > 1$.

If the resultant orbit is not elliptical, then a different method for determining the retrograde impulse is necessary. Although it is not envisaged that it will ever be desirable to arrive at a certain pericynthion position on a hyperbolic trajectory, it is possible to determine the necessary retrograde impulse vector which will accomplish this. If equation (2.9) is modified to

$$V_R^2 = V_A^2 + A^2 + K + 2V_A \phi A X^{\frac{1}{2}} \tag{2.11}$$

then solution of equation (2.11) will yield the necessary velocity. The orbit velocity vector in this case will be opposite in direction from that which would be determined if equation (2.9) were utilized.

To obtain an insight into the quantitative values of retrograde velocity, refer to figure 4 and the orbit reference table. Figure 4 shows the retrograde velocities necessary for various combinations of arrival and required conditions and the orbit reference table defines the orbits established as a result of these retrograde impulses. As an example, suppose it is desired to establish an orbit with a pericynthion altitude of 100 nautical miles with the pericynthion radius orientated 150° from the insertion radius. Assume that the vehicle arrives at an altitude of 1,000 nautical miles with a velocity of 6,500 ft/sec. Reference to figure 4 shows that this will require a retrograde impulse of 3,360 ft/sec. Reference to the orbit reference table, condition 2-B, shows that the established orbit will have an apocynthion altitude of 1,128.6 nautical miles, an apocynthion velocity of 3,031.5 ft/sec, a pericynthion velocity of 6,036.9 ft/sec, and the time required for one complete orbit will be 3.8 hours.

It has been determined that the preceding equations will yield the retrograde impulse for any specified values of pericynthion radius and orientation. However, it is possible in many cases to reach these pericynthion altitudes with smaller retrograde velocities if the restrictions on the orientation of pericynthion are relaxed. Upon referring to equation (2.9), it is seen that the quantities A, K, and ϕ are independent of θ, therefore differentiation of equation (2.9) with respect to θ yields

$$\frac{\partial V_R^2}{\partial \theta} = A \left(A - V_A \phi X^{-\frac{1}{2}} \right) \frac{\partial X}{\partial \theta} \tag{2.12}$$

and differentiation of identity (c) yields)

$$\frac{\partial X}{\partial \theta} = \frac{\left(1 - \frac{1}{\phi}\right)\sin\theta}{\left(1 - \frac{\cos\theta}{\phi}\right)^2} \tag{2.13}$$

To obtain a minimum retrograde velocity then $\frac{\partial V^2}{\partial \theta} = 0$, and

$$\frac{\partial V_R^{\,2}}{\partial \theta} = A\left(A - V_A \phi\, X^{-\frac{1}{2}}\right)\left[\frac{\left(1 - \frac{1}{\phi}\right)\sin\theta}{\left(1 - \frac{\cos\theta}{\phi}\right)^2}\right] = 0 \tag{2.14}$$

It is known that $A \neq 0$ therefore

$$\frac{\partial V_R^{\,2}}{\partial \theta} = \left(A - V_A\, \phi\, X^{-\frac{1}{2}}\right)\left[\frac{\left(1 - \frac{1}{\phi}\right)\sin\theta}{\left(1 - \frac{\cos\theta}{\phi}\right)^2}\right] = 0 \tag{2.14-A}$$

Suppose tentatively that for equation (2.14-A) to hold that

$$A - V_A\, \phi\, X^{-\frac{1}{2}} = 0 \tag{2.15}$$

therefore

$$X^{-1} = \frac{A^2}{\left(V_A\, \phi\right)^2} \tag{2.15-A}$$

Substitution of equation (2.15-A) into identity (c) yields

$$\text{Cos}\ \theta = \frac{1 - \left(\frac{A}{V_A\, \phi}\right)^2}{\frac{1}{\phi} - \left(\frac{A}{V_A \phi}\right)^2} \tag{2.16}$$

Equation (2.16) has two boundary conditions

$$\theta = 0^{\circ} \quad \text{and} \quad \theta = -180^{\circ}$$

When $r = r_p$, then $\theta = 0$ by definition of the two-body orbit and this is the unique value of θ to be considered.

By reference to equation (2.14-A), it is seen that $\theta = -180^{\circ}$ is an alternative solution to

$$\frac{\partial V_R^{\,2}}{\partial \theta} = \frac{\left(1 - \frac{1}{\phi}\right) \sin \theta}{\left(1 - \frac{\cos \theta}{\phi}\right)^2} = 0 \tag{2.17}$$

Two conditions are now stipulated:

Condition (1) where

$$\theta = -180^{\circ} \quad \text{and} \quad \frac{\partial V_R^{\,2}}{\partial \theta} = 0 \quad \text{let} \quad V_R = V_{R1}$$

Condition (2) where

$$-180^{\circ} \leqslant \theta \leqslant 0 \quad \text{and} \quad \frac{\partial V_R^{\,2}}{\partial \theta} = 0 \quad \text{let} \quad V_R = V_{R2}$$

The following analysis is to prove that when V_{R2} exists, it is a minimum. Tentatively assume that $V_{R1}^{\,2} \geqslant V_{R2}^{\,2}$ in which case the following is evolved from equation (2.9).

$$A^2 X_1 - 2V_A \, \phi \, A \, X_1^{\frac{1}{2}} \geqslant A^2 X_2 - 2V_A \, \phi \, A \, X_2^{\frac{1}{2}} \tag{2.18}$$

It is convenient to add $2V_A \, \phi \, AX_1^{\frac{1}{2}} - A^2 X_2$ to both sides. This allows inequality (2.18) to be rewritten

$$A^2\left(X_1^{\frac{1}{2}} + X_2^{\frac{1}{2}}\right)\left(X_1^{\frac{1}{2}} - X_2^{\frac{1}{2}}\right) \geq 2V_A \phi A\left(X_1^{\frac{1}{2}} - X_2^{\frac{1}{2}}\right) \tag{2.18-A}$$

By definition $X = \dfrac{1 - \cos\theta}{1 - \frac{1}{\phi}\cos\theta}$

therefore

$$X_1^{\frac{1}{2}} = \left(\frac{2}{1 + \frac{1}{\phi}}\right)^{\frac{1}{2}} \tag{2.19}$$

For condition (2) equation (2.17) yields

$$-1 \leqslant \frac{1 - \left(\frac{A}{V_A \phi}\right)^2}{\frac{1}{\phi} - \left(\frac{A}{V_A \phi}\right)^2} \tag{2.20}$$

and substituting the value of A^2 from equation (2.16), inequality (2.20) simplifies to

$$X_2 \geqslant \frac{2}{1 + \frac{1}{\phi}} \tag{2.21}$$

and

$$X_2^{\frac{1}{2}} \geqslant \left(\frac{2}{1 + \frac{1}{\phi}}\right)^{\frac{1}{2}} \tag{2.21-A}$$

From equation (2.19) and inequality (2.21-A) it follows that

$$X_1^{\frac{1}{2}} - X_2^{\frac{1}{2}} \leqslant 0 \tag{2.22}$$

Inequality (2.18-A) which is the condition for $V_{R1}^2 \geqslant V_{R2}^2$ and states that V_{R2} is a minimum, is now divided by $\left(X_1^{\frac{1}{2}} - X_2^{\frac{1}{2}}\right)A^2$, a negative number which reverses the inequality.

$$X_1^{\frac{1}{2}} + X_2^{\frac{1}{2}} \leqslant \frac{2V_A\phi}{A} \tag{2.23}$$

From equation (2.15-A), both sides of which are positive, it is seen that

$$\frac{2V_A\phi}{A} = 2X_2^{\frac{1}{2}} \tag{2.24}$$

therefore substitution of equation (2.24) into inequality (2.23) yields

$$X_1^{\frac{1}{2}} + X_2^{\frac{1}{2}} \leqslant 2X_2^{\frac{1}{2}} \tag{2.25}$$

Subtraction of $2X_2^{\frac{1}{2}}$ from both sides of inequality (2.25) yields

$$X_1^{\frac{1}{2}} - X_2^{\frac{1}{2}} \leqslant 0 \tag{2.26}$$

but from inequality (2.22), it is known that this is true, therefore, when V_{R2} exists, it is a minimum. It must be noted that V_{R2} is a minimum for a conic trajectory. For the trajectory to be elliptical, two-body orbital equations yield equation (2.5) and for elliptical conditions to exist

$$2\frac{rp}{r} - 1 - \cos\theta > 0$$

or

$$2\frac{rp}{r} - 1 > \cos\theta \tag{2.27}$$

Hence when V_{R2} exists and $2\frac{rp}{r} -1 > \cos \theta$, then it is the minimum retrograde velocity for an elliptical orbit.

In summation:

If V_{R2} does exist, it is a minimum where

$$\frac{\partial V_R^{\ 2}}{\partial \theta} = 0$$

and

$$-1 \leqslant \frac{1 - \left(\frac{A}{V_A \phi}\right)^2}{\frac{1}{\phi} - \left(\frac{A}{V_A \phi}\right)^2} \leqslant 1$$

when $\phi = 1$, $\theta = 0$ is the unique value which satisfies equation (2.14). If V_{R2} does not exist and $\phi \neq 1$, then V_{R1} is a minimum. If V_{R1} is a minimum, $\theta = -180^{\circ}$ and

$$\frac{\partial V_R^{\ 2}}{\partial \theta} = 0$$

and if

$$2\frac{rp}{r} -1 > \cos \theta$$

then the orbit is elliptical.

The method for determining the minimum retrograde velocity for a given set of conditions is now outlined.

Step 1

Determine from the following equation if V_{R2} does exist:

$$\theta = \cos^{-1}\left[\frac{1 - \left(\frac{A}{V_A\phi}\right)^2}{\frac{1}{\phi} - \left(\frac{A}{V_A\phi}\right)^2}\right] \tag{2.28}$$

If there is a solution to equation (2.28), then V_{R2} does exist and the determined value of θ will yield the minimum retrograde impulse. Should the equation have no legitimate solution, then V_{R2} does not exist and therefore, $\theta = -180^o$, which is the value of θ for V_{R1}, will yield the minimum retrograde impulse.

Step 2

The eccentricity of the orbit should now be determined. By using the value of θ as found from equation (2.28), the eccentricity can be determined from

$$e = \frac{X}{\phi} - 1 \tag{2.29}$$

Step 3

The minimum retrograde velocity for an elliptical orbit can be ascertained by using the determined value of θ in equation (2.9) which is

$$V_R^2 = V_A^2 + A^2X + K - 2V_A\phi AX^{\frac{1}{2}} \tag{2.9}$$

If the resultant orbit is not elliptical, the minimum retrograde velocity can be determined from equation (2.11) which is

$$V_R^2 = V_A^2 + A^2X + K + 2V_A\phi AX^{\frac{1}{2}} \tag{2.11}$$

A comparison between minimum retrograde velocities and retrograde velocities for constrained conditions is shown in figure 5. For the construction of this figure, the vehicle is assumed to arrive at an altitude of 1,000 nautical miles with an arrival velocity of 7,000 ft/sec. Graph (A) shows $\left(\frac{rp}{r}\right)$, where r is the arrival radius, plotted against θ which will yield the minimum retrograde velocity. Graph (B) shows $\left(\frac{rp}{r}\right)$ plotted against the required minimum retrograde velocity and the comparison curve shows the retrograde velocities which yield a pericynthion orientation of 150^o. Graph (C) shows $\left(\frac{rp}{r}\right)$ plotted against the resultant orbit apocynthion radius for the case where the retrograde impulse is a minimum and the comparison case. It can be seen that over a considerable range of $\left(\frac{rp}{r}\right)$, the minimum retrograde impulse will offer some fuel savings without influencing the resultant orbits significantly. In the range of $\left(\frac{rp}{r}\right)$ where the fuel savings are considerable, however, the resultant orbits have the following undesirable features: large apocynthion radii, large pericynthion velocities, and large orbit periods.

Another example of retrograde determination is shown in figure 6. Graph (b) of this figure shows the variation of arrival velocity as a function of altitude for a given earth insertion velocity and position. Graph (a) shows the retrograde impulses necessary to establish orbits of various configurations for the arrival velocities and altitudes shown in graph (b). Although it is stipulated that the pericynthion radius be orientated 180^o from the arrival radius, it should be noted that the constraints for these particular cases allow a minimum retrograde impulse. This can be confirmed by substituting values into equation (2.28). It should also be mentioned that since $\theta = 180^o$, the arrival altitude becomes the apocynthion altitude of the established orbit. This figure also shows that the nearer the required orbit is to a circular orbit, the lower the required retrograde velocity. Also, for a low altitude survey, say from an altitude of 100 nautical miles, the nearer the orbit is to circular, the smaller the pericynthion velocity will be, which will allow a longer survey time.

Figure 6(c) is introduced to extend the quantitative information included in this report. The classical restricted three-body model referred to on page 7 yields a relationship between earth insertion (V) and arrival (V). Examination of Jacobian constant contours reveals that there is absolute maximum value of C and a corresponding minimum velocity for a given earth insertion position vector which could possibly allow the lunar vehicle to free coast to the moon. This maximum value of C and corresponding minimum velocity is taken as a convenient mathematical base. The reference minimum velocity in figure 6(c) is that for an altitude of 100 nautical miles, moon lead angle of 90^o and an earth-moon distance of 192,358 nautical miles. The associated velocity for a given C will vary with position according to equation (1.9), thus figure 6(c) affords a study of the consequences of a variety of earth insertion conditions.

FUEL REQUIREMENTS

The fuel required for instantaneous retrograde impulse for establishing lunar orbits can be calculated from the equation

$$\delta = 1 - e^{\frac{-V_R}{Ig_o}} \tag{3.1}$$

where δ is the ratio of the fuel mass at commencement of burning to the gross mass. Shown in figure 7 are the fuel mass to gross mass ratios necessary for establishing the orbits shown in figure 6. The fuel consumption values as shown are considered absolute minimums since the impulses are assumed to be instantaneous. Fuel consumption values for corresponding finite impulses, however, will differ little from the given values. In all cases where thrust is acting against a resolved weight component, there is a loss in efficiency and this to a small degree would be the case with corresponding impulses of finite duration.

As an example of the use of this graph, consider a lunar vehicle with an earth surface weight of 10,000 pounds containing a fuel of 250 seconds specific impulse. From the graph it will be noted that, within the conditions considered, the earth surface weight of fuel required for inserting the vehicle at a lunar altitude of 5,000 nautical miles into an orbit with a pericynthion altitude of 100 nautical miles is 3,200 pounds.

To obtain the total fuel requirement for orbit entry and exit, the following is considered. The retrograde velocity to insert into the orbit is assumed to equal the posigrade velocity for exit. This assumption is made since the velocity requirements relative to the center of the moon of the major earth-moon trajectory will not have substantially changed after a restricted number of lunar orbits. The total fuel requirement is found from the equation

$$\bar{\delta} = 1 - e^{\frac{-V_R}{Ig_o}} \tag{3.2}$$

or

$$\bar{\delta} = 2\delta - \delta^2 \tag{3.3}$$

where $\bar{\delta}$ is the total fuel mass to gross mass ratio required for orbit entry and exit. Shown in figure 8 are the total fuel mass to gross mass ratios necessary for orbit entry and exit for the orbits shown in figure 6. As an example of the use of this graph, consider a lunar vehicle with an earth surface weight of 10,000 pounds containing a fuel of 250 seconds specific impulse. From the graph, it can be seen that the earth surface weight of fuel required for both entry and exit, at an altitude of 5,000 nautical miles for an orbit with a pericynthion altitude of 100 nautical miles, is 5,400 pounds.

DISCUSSION ON ORBIT STABILITY

The simplest and possibly the best method of considering the stability of the vehicle's orbit about the moon is to consider the vehicle as a satellite of the moon where the following are considered the major perturbation effects:

(a) the earth's gravity field

(b) the sun's gravity field

(c) the moon's potential distribution

(d) the lunar librations

By stability, it is implied that successive orbits have repetitive characteristics.

A brief analysis of the individual gravitational effects of the earth, sun, and moon show the possibility of a highly stable orbit. The order of magnitude of the gravitational effects of the earth, sun, and moon on the vehicle are shown below.

	Zero lunar altitude	Lunar altitude 5,000 n.m.
Moon	5.31	0.13 ft/sec^2
Sun	0.02	.02 ft/sec^2
Earth	.01	.01 ft/sec^2

If for a short duration there is insignificant difference between the effects on the moon and the effects on the vehicle due to the gravity fields of the earth and sun, then the moon and vehicle will tend to behave as a two-body system with a resultant stable orbit. Should the apocynthion altitude of the vehicle's orbit never be greater than 5,000 nautical miles, then the scalar accelerations of the vehicle and moon due to the sun never have a greater ratio than 1.000147 or the inverse. The directional difference in the acceleration vectors is negligible since the sun is approximately 80,764,000 nautical miles away. The corresponding ratio due to the earth is never greater than 1.0647 or the inverse and the directional difference is small. Viewed in this manner, the sun's gravitational effect is very small. The moon and vehicle acceleration vectorial difference due to the earth appears more significant, but due to the oscillatory nature of the difference, since it is periodic with the vehicle's orbit, the effect is small over a restricted number of orbits.

Another perturbation effect of interest is the librations of the moon about its center of gravity. The apparent librations as viewed from earth are of no concern in this study. There is a small real libration in longitude due to the eccentricity of the moon's orbit about the barycenter but the period is a month, and hence this libration does not present a problem. If any high frequency librations exist, they are thought to be insignificantly small.

The main changes in orbit characteristics with time are anticipated to be changes in the inclination of the orbit and regression of the nodes relative to the lunar equator.

Several stability checks have been conducted by a comprehensive simulation incorporated in a digital mathematical model which includes earth, sun, and lunar potential distribution effects. (See ref. 1.)

This model is particularly attractive in that the origin of integration is the moon for the conditions considered and the round-off errors are those involved in the integration of perturbations from reference ellipses. The checks added confidence to the above remarks concerning stability.

CONCLUSIONS

1. At low lunar altitudes, say less than 5,000 nautical miles, the lunar vehicle in free-coast conditions will essentially behave as a satellite of the moon, and hence the trajectory will be conical with the center of the moon mass as a focal point. Depending on the velocity imparted by the retrograde impulse, the trajectory will be elliptic, parabolic, or hyperbolic. Elliptical trajectories are of interest in that closed orbits about the moon are required for survey purposes.

2. If the retrograde impulse is such that the resultant orbit will be elliptical, a simple thrusting logic may be introduced for obtaining a required pericynthion position, the only requirement being that the insertion velocity is that which would be yielded by the classical two-body solution for the required conditions.

3. The lunar arrival velocity increases with increase in earth-insertion velocity and this entails heavier fuel expenditure for insertion into a local lunar orbit; however, there is the possibility that the trajectory associated with a higher earth-insertion velocity will require less fuel expenditure for guidance before the arrival phase.

4. For a given arrival altitude, the nearer the required orbit is to a circular orbit, the lower the required retrograde velocity. Furthermore, for a given pericynthion altitude, the nearer the associated orbit is to circular, the smaller the pericynthion velocity. The pericynthion is probably above the center of the area to be surveyed, and hence a small pericynthion velocity is desirable.

5. For an elliptical orbit with pericynthion altitude and orientation stipulated, there is a unique retrograde velocity. This retrograde velocity may or may not be the minimum which will yield the desired pericynthion altitude. If the restriction on the orientation of the pericynthion is relaxed, then in many cases it is possible to determine a smaller retrograde impulse which will yield the desired pericynthion altitude. Utilization of the minimum retrograde impulse, however, may produce an orbit with an excessive apocynthion altitude, pericynthion velocity, and period. It is possible to minimize these adverse features, at the expense of fuel, by orientating the pericynthion gradually away from the point of minimum retrograde impulse to a point where the orbit characteristics become more desirable.

APPENDIX

TWO-BODY EQUATIONS

In order to maintain continuity in the main text, transposes of well-known two-body relationships are immediately used. This appendix is included to indicate the derivation of these relationships.

The differential equations of motion are

$$M\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = -\frac{GMm}{r^2} \qquad \text{(A-1)}$$

and

$$\frac{M}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \qquad \text{(A-2)}$$

From these equations and a knowledge of conic geometry, the following equations can be evolved: The general conic expression for the semilatus rectum is

$$p = r(1 + \epsilon \cos \theta) \qquad \text{(A-3)}$$

and when $r = r_p$, then $\theta = 0$ which yields

$$p = r_p(1 + \epsilon) \qquad \text{(A-4)}$$

The expression for the angle γ which is the angle between the velocity vector V_o and the normal to the vector r is from integration of equation (A-2) above

$$\cos \gamma = \frac{r_p V_{op}}{rV_o} \qquad \text{(A-5)}$$

It can be seen that equation (A-5) is the same as equation (2.2) of the main text.

The expression for the semimajor axis which is evolved from equation (A-3) is

$$a = \frac{p}{1 - \epsilon^2} \tag{A-6}$$

Substitution of the values of p and ϵ from equations (A-3) and (A-4) yield

$$a = \frac{r_p\left(\frac{r_p}{r} - \cos\theta\right)}{2\frac{r_p}{r} - 1 - \cos\theta} \tag{A-7}$$

It can be seen that equation (A-7) is the same as equation (2.5) of the main text. For a hyperbola since $\epsilon > 1$ it can be seen from equation (A-6) that the numerical value of a will be negative.

The orbit velocity is

$$V_o^2 = GM_o\left(\frac{2}{r} - \frac{1}{a}\right) \tag{A-8}$$

Substituting into equation (A-8), the value of a of equation (A-7) yields

$$V_o^2 = \frac{GM_o}{r_p}\left[\frac{r}{r_p}\left(\frac{1 - \cos\theta}{1 - \frac{r}{r_p}\cos\theta}\right) + 2\left(\frac{r_p}{r} - 1\right)\right] \tag{A-9}$$

This is the same as equation (2.7) of the main text. The pericynthion velocity is

$$V_{op}^2 = GM_o\left[\frac{2}{r_p} - \frac{1}{a}\right] \tag{A-10}$$

Substitution of the value of a from equation (A-7) yields

$$V_{op}^2 = GM_o\left[\frac{1 - \cos\theta}{r_p\left(\frac{r_p}{r} - \cos\theta\right)}\right] \tag{A-11}$$

It can be seen that equation (A-11) is the same as equation (2.6) of the

REFERENCES

1. Republic Aircraft Corp. Program. Interplanetary Trajectories by the Encke Method Programmed for IBM 704, 1959.

2. Buchheim, R. W.: Lunar Flight Trajectories. Rand Corporation Report No. P-1268. Jan. 30, 1958.

3. Moulton, F. R.: Celestial Mechanics. MacMillan Publishing Co. Apr. 1914.

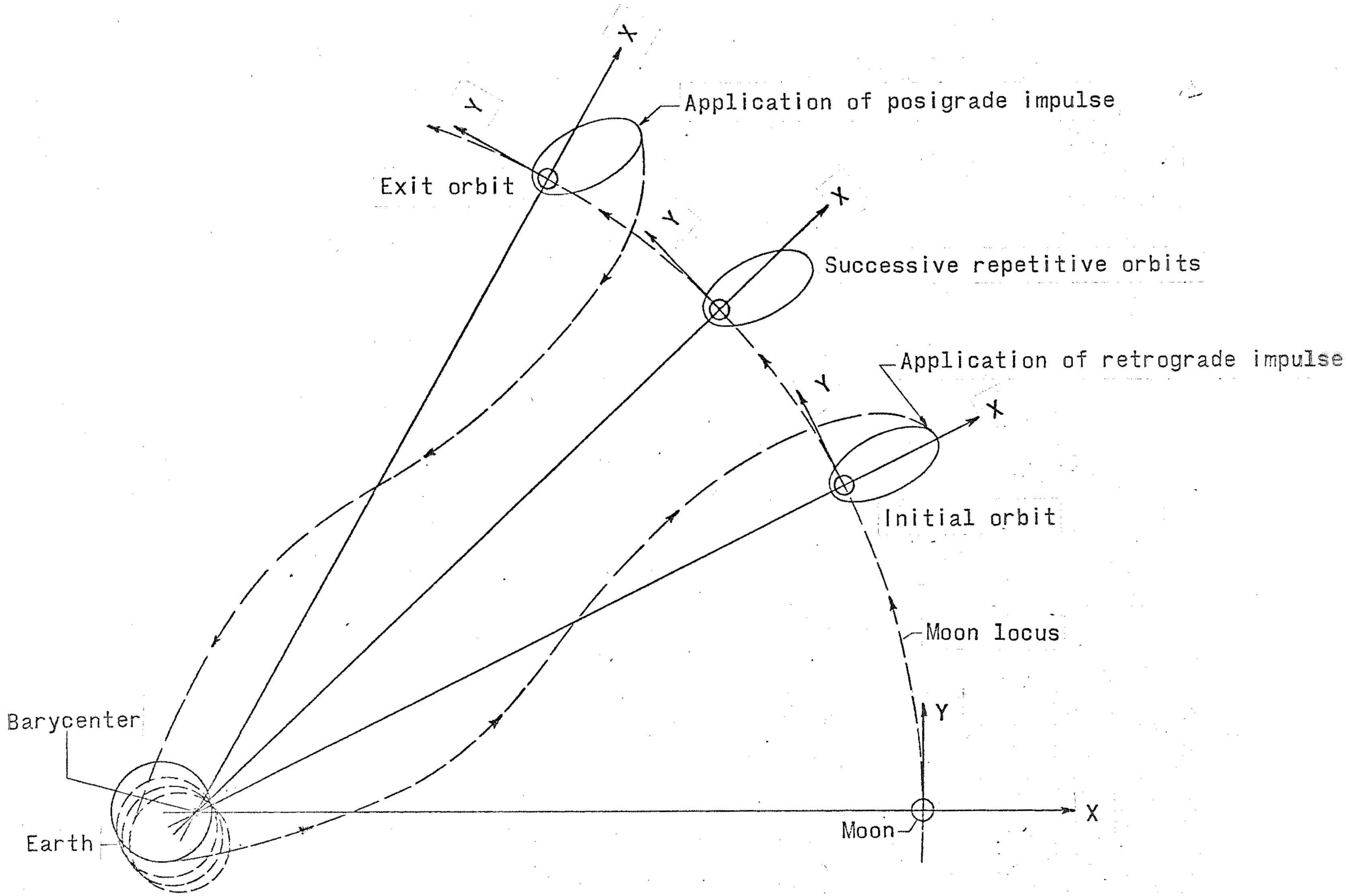


Figure 1.- Diagram of envisaged manned lunar mission.

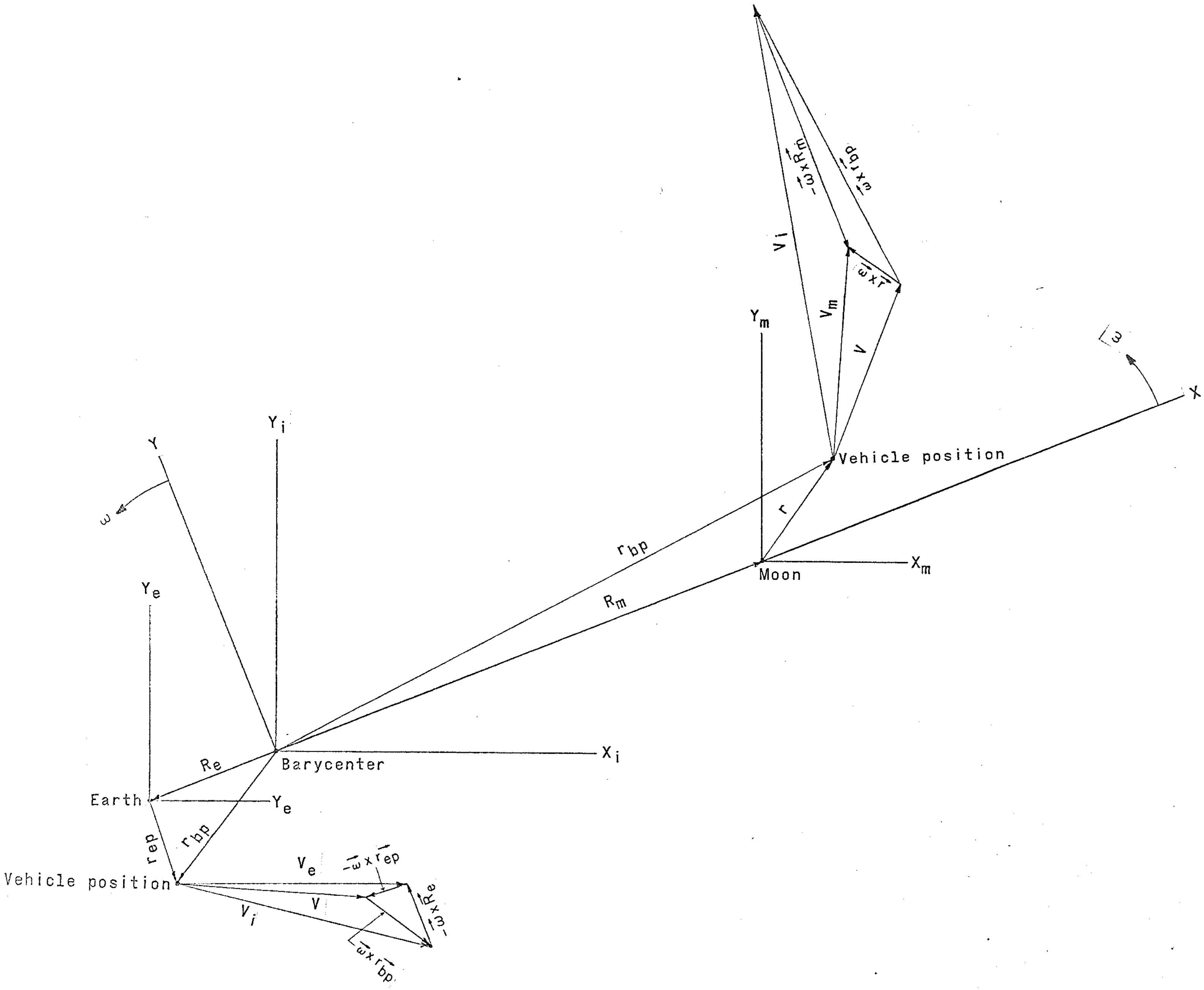


Figure 2.- The relationship of velocity vectors.

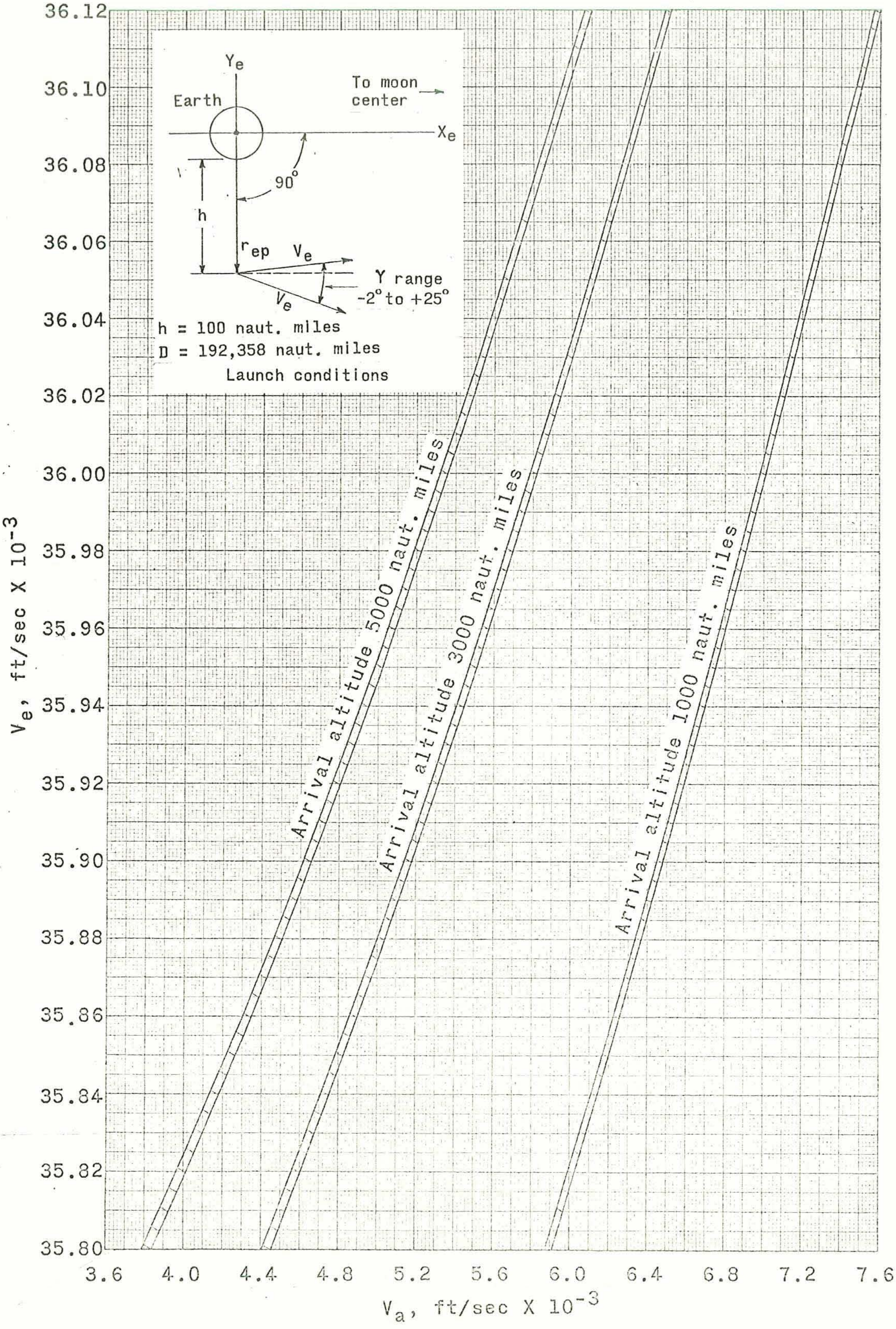


Figure 3.- Variation of earth insertion velocity (V_e) with lunar arrival velocity (V_a) as determined by Jacobian relationship.

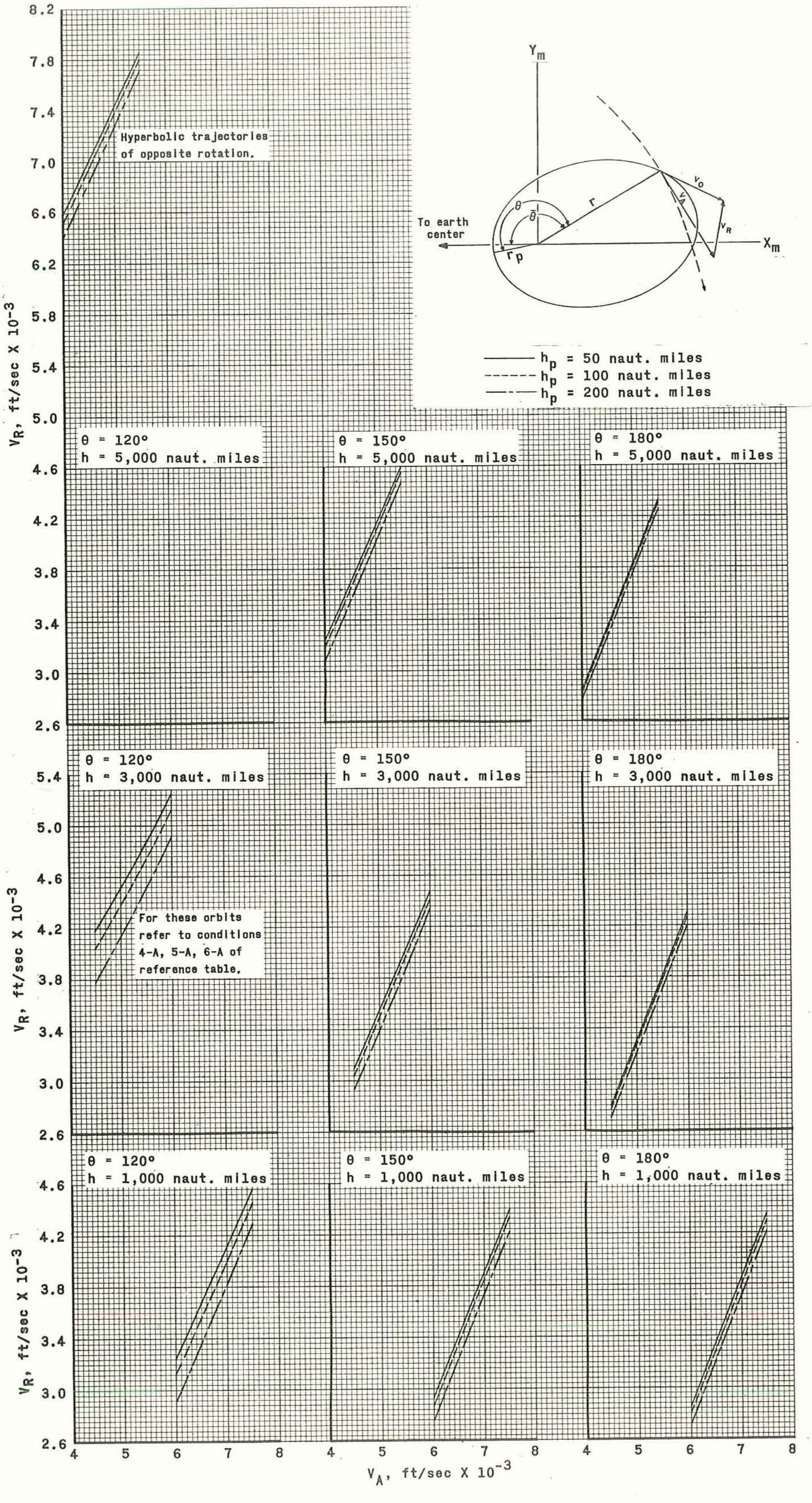


Figure 4.- Retrograde velocities for required pericynthion vectors.

CONDITION			A	B	C
	ARRIVAL ALTITUDE - 1,000 NAUT. MILES				
		h_p, naut. miles	$\theta = 120^\circ$	$\theta = 150^\circ$	$\theta = 180^\circ$
1	h_a, naut. miles	50	1,914.0	1,143.6	1,000.0
	V_{oa}, ft/sec		2,263.8	2,963.8	3,146.1
	V_{op}, ft/sec		6,535.5	6,244.9	6,171.7
	Period, hours		5.3	3.8	3.5
2	h_a	100	1,788.0	1,128.6	1,000.0
	V_{oa}		2,397.1	3,031.5	3,197.5
	V_{op}		6,296.8	6,036.9	5,970.9
	Period		5.1	3.8	3.6
3	h_a	200	1,593.0	1,102.9	1,000.0
	V_{oa}		2,638.5	3,156.7	3,293.2
	V_{op}		5,868.9	5,661.8	5,608.7
	Period		4.9	4.0	3.8
	ARRIVAL ALTITUDE - 3,000 NAUT. MILES				
		h_p, naut. miles	$\theta = 120^\circ$	$\theta = 150^\circ$	$\theta = 180^\circ$
4	h_a	50	* 844,339.7	4,074.3	3,000.0
	V_{oa}		8.8	1,366.0	1,701.0
	V_{op}		7,578.9	6,931.0	6,780.4
	Period		17,330.2	10.3	7.7
5	h_a	100	* 56,562.3	3,988.0	3,000.0
	V_{oa}		132.3	1,416.6	1,734.6
	V_{op}		7,333.1	6,724.4	6,582.3
	Period		315.3	10.2	7.8
6	h_a	200	* 20,965.9	3,845.1	3,000.0
	V_{oa}		357.9	1,510.8	1,798.3
	V_{op}		6,889.2	6,350.8	6,223.8
	Period		77.9	10.1	8.0
	ARRIVAL ALTITUDE - 5,000 NAUT. MILES				
		h_p, naut. miles	$\theta = 120^\circ$	$\theta = 150^\circ$	$\theta = 180^\circ$
7	h_a	50	Hyperbolic	8,335.5	5,000.0
	V_{oa}		Trajectory	767.9	1,168.2
	V_{op}			7,209.1	7,021.7
	Period			23.1	12.8
8	h_a	100	Hyperbolic	8,044.8	5,000.0
	V_{oa}		Trajectory	809.3	1,193.0
	V_{op}			7,005.4	6,826.5
	Period			22.3	13.0
9	h_a	200	Hyperbolic	7,578.9	5,000.0
	V_{oa}		Trajectory	886.6	1,240.4
	V_{op}			6,636.5	6,472.8
	Period			21.1	13.2

*In the solar system the assumption of elliptical characteristics in these cases is not valid.

Orbit Reference Table

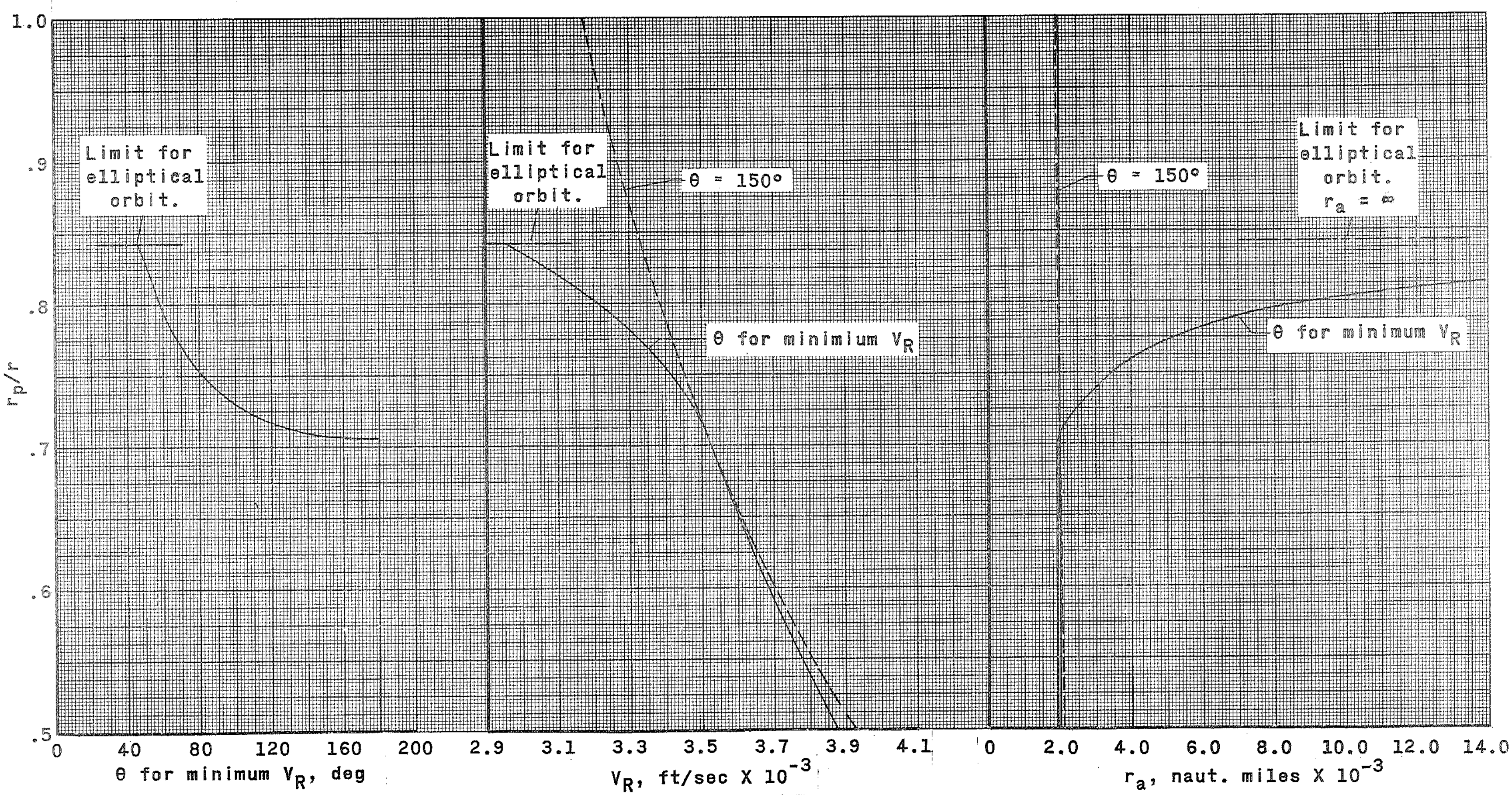
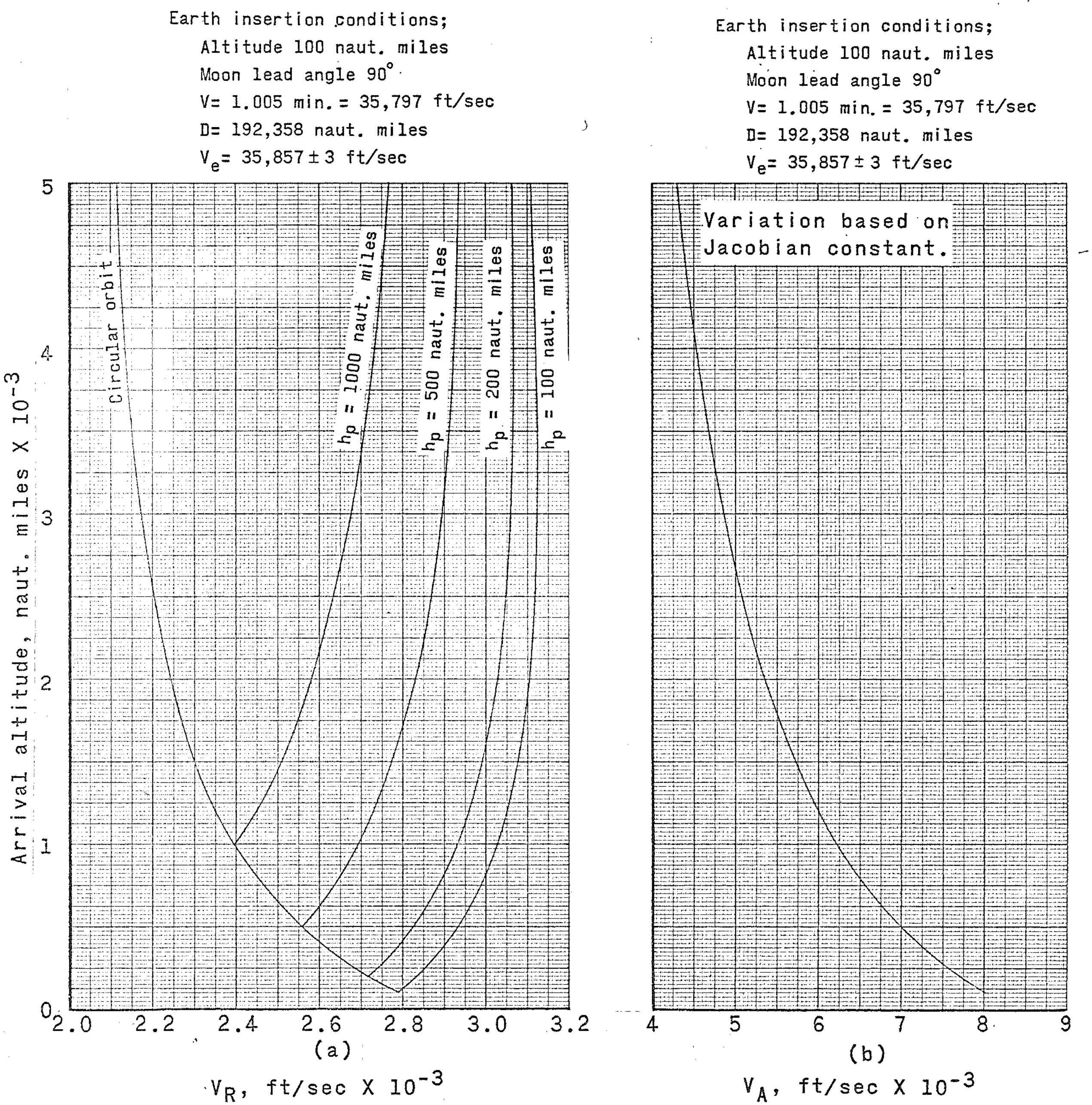


Figure 5.- Comparison of retrograde velocities for minimum and constrained conditions. V_a = 7,000 ft/sec; h = 1,000 naut. miles.



Figures 6a and 6b.- Retrograde velocity for insertion into orbit for θ = 180°.

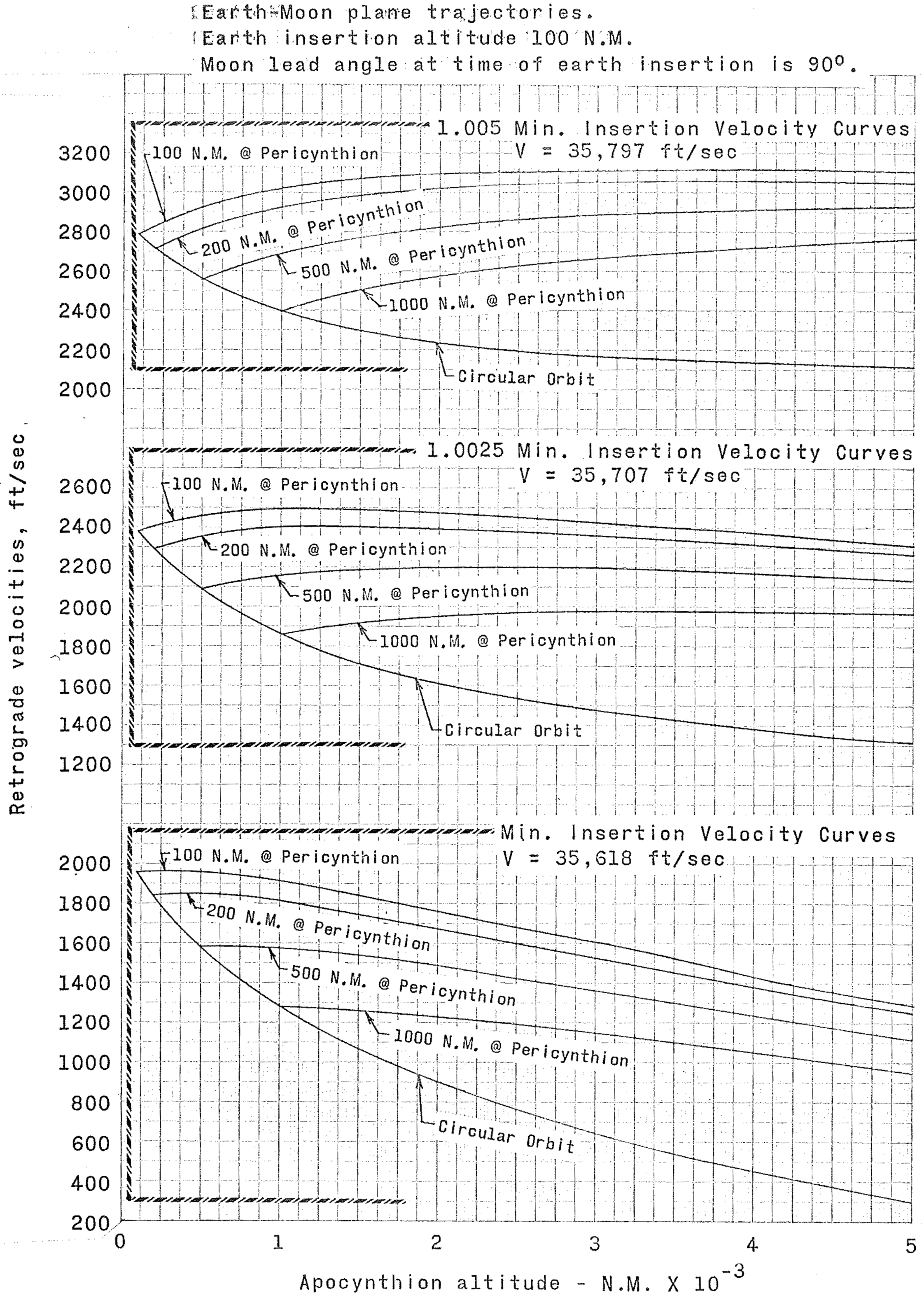


Figure 6c.- Retrograde velocity related to earth insertion conditions in terms of the Jacobian relationship. $\theta = 180^{\circ}$.

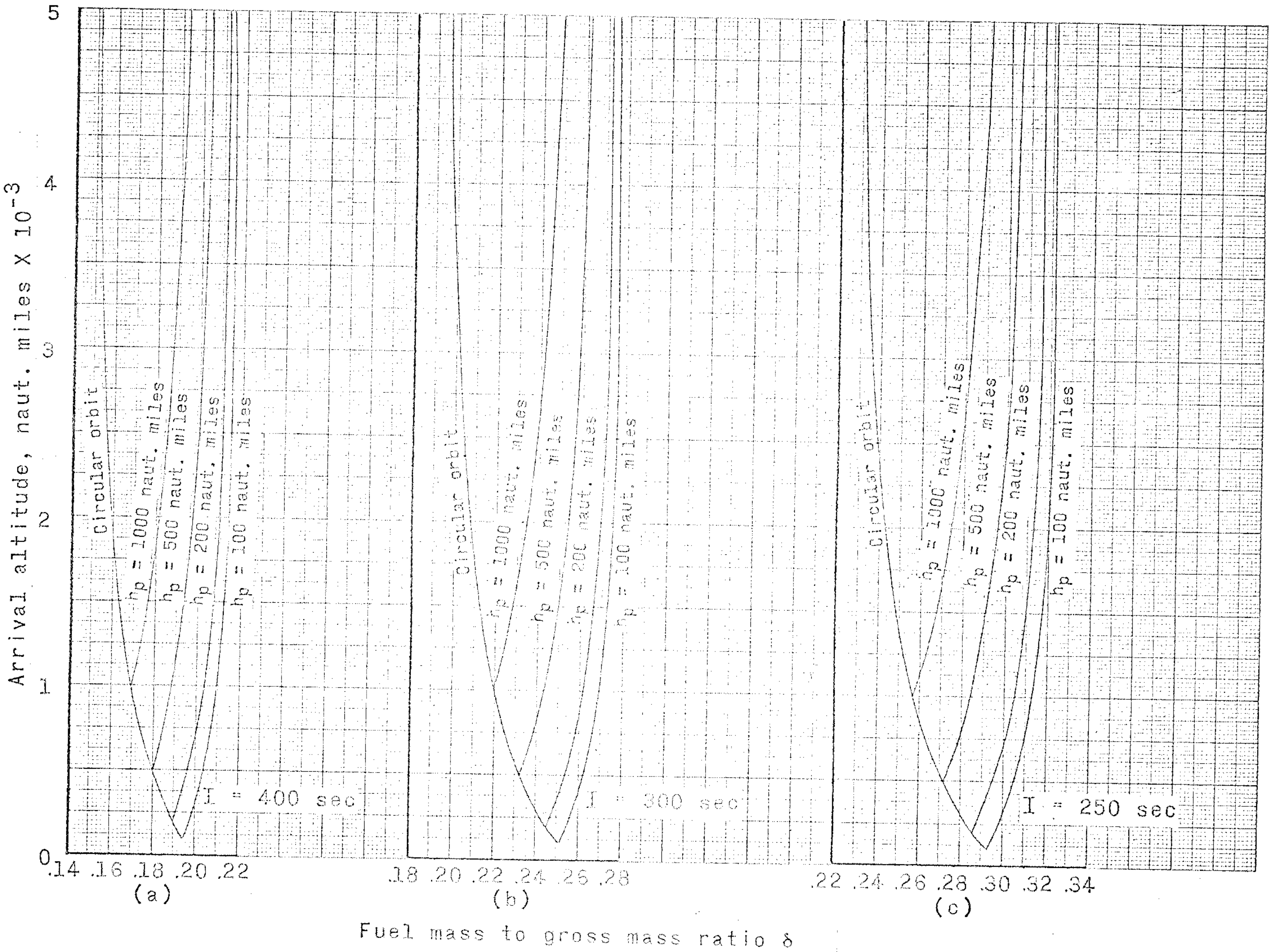


Figure 7.- Fuel consumption for insertion into orbit.

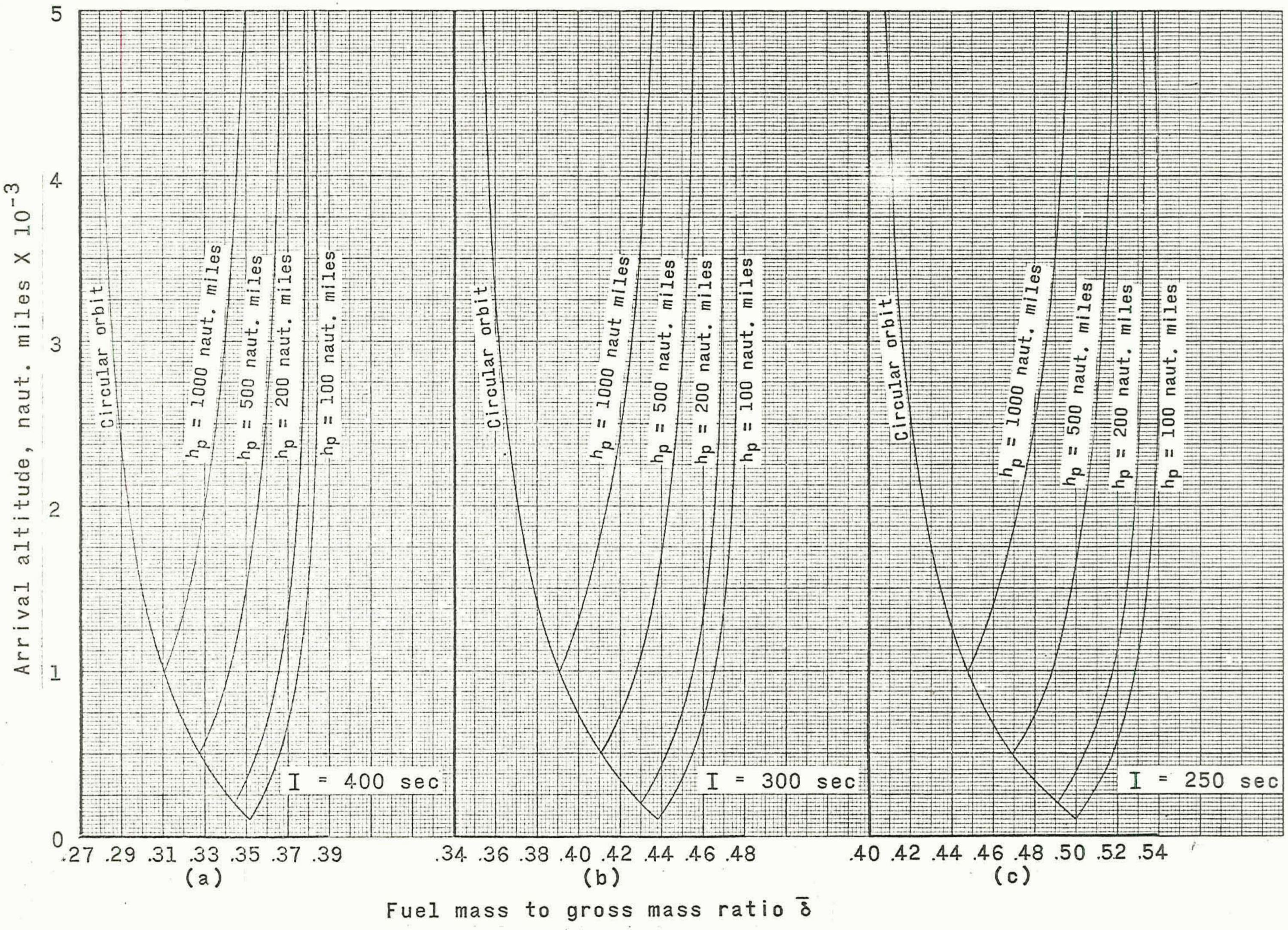


Figure 8.- Total fuel consumption for orbit entry and exit.